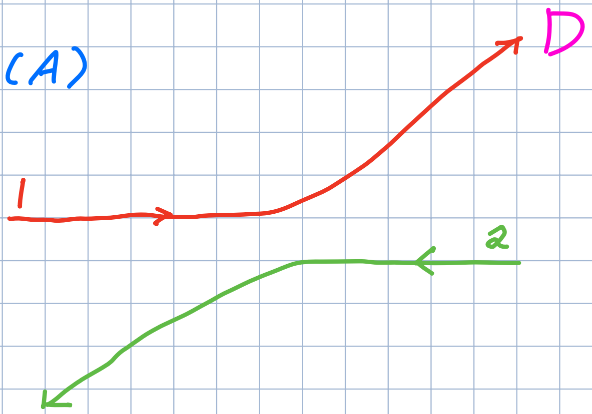
Identical Particles

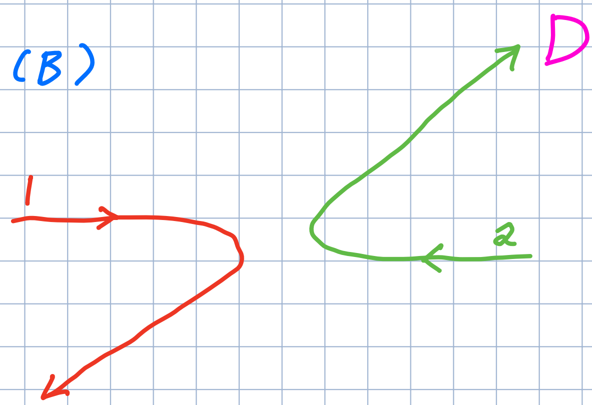
No experiment can tell one from another

Example: Scattering: two trajectories $\alpha = A, B$

(A)



(B)



Classically can distinguish trajectory A from B: $P_D = \sum_\alpha P_\alpha$

QM cannot distinguish $\Rightarrow$ $P_D = \left| \sum_\alpha A_\alpha \right|^2$ (inteference)

Example: Exchange degeneracy

Tensor product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ $\dim(\mathcal{H}) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$

e.g. two different spin ½: $|1:+, 2:-\rangle \neq |2:+, 1:-\rangle$ (e.g. electron and proton)

But for identical particles $|1:+, 2:-\rangle \equiv |2:+, 1:-\rangle$

cannot be distinguished

Different ket vectors but no measurement can discriminate

$\Rightarrow$ Same physical state

$\therefore$ Tensor product basis contains redundant states

Need new postulate to remove redundancy

Permutation Group

2 particle permutations

transpositions

distinguishable but isomorphic state spaces

$\mathcal{H}_1 \sim \mathcal{H}_2$ $\{|u_i\rangle\}$ is basis for $\mathcal{H}_1$ and $\mathcal{H}_2$

Tensor product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ basis $\{|1{:}u_i \; 2{:}u_j\rangle\}$

Note: $|1{:}u_i \; 2{:}u_j\rangle \equiv |2{:}u_j \; 1{:}u_i\rangle \neq |1{:}u_j \; 2{:}u_i\rangle$ if $i \neq j$

Permutation $P_{21} \; |1{:}u_i \; 2{:}u_j\rangle \equiv |2{:}u_i \; 1{:}u_j\rangle = |1{:}u_j \; 2{:}u_i\rangle$

$(P_{21})^2 = 1$ involution

$P_{21}^\dagger = P_{21}$ hermitian

$P_{21}^\dagger P_{21} = P_{21} P_{21}^\dagger = 1$ unitary

Projectors: $S \equiv \frac{1}{2}(1 + P_{21})$ $\quad S^2 = S$ $\quad P_{21} S = S$ $\quad S + A = 1$

$A \equiv \frac{1}{2}(1 - P_{21})$ $\quad A^2 = A$ $\quad P_{21} A = -A$ $\quad SA = AS = 0$

(Anti) Symmetric subspace

$|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$P_{21}(S|\psi\rangle) = (S|\psi\rangle) \implies S|\psi\rangle \in \mathcal{H}_S \subset \mathcal{H}$

$P_{21}(A|\psi\rangle) = -(A|\psi\rangle) \implies A|\psi\rangle \in \mathcal{H}_A \subset \mathcal{H}$

$\mathcal{H}_S \cup \mathcal{H}_A = \mathcal{H}$

Example: two spin ½'s

Basis for $\mathcal{H}_S$: $\{|1{:}+ \; 2{:}+\rangle, \; |1{:}- \; 2{:}-\rangle, \; \frac{1}{\sqrt{2}}(|1{:}+ \; 2{:}-\rangle + |1{:}- \; 2{:}+\rangle)\}$ dim 3

$\mathcal{H}_A$: $\{\frac{1}{\sqrt{2}}(|1{:}+ \; 2{:}-\rangle - |1{:}- \; 2{:}+\rangle)\}$ dim 1

Transformation of observables

let $\{|u_i\rangle\}$ be eigenstates of $B(1)$ in $\mathcal{H}_1$

eigenvalues $\{b_i\}$

$$(P_{21} B(1) P_{21}^\dagger)\,|1{:}u_i \; 2{:}u_j\rangle = P_{21} B(1)\,|1{:}u_j \; 2{:}u_i\rangle$$

$$= b_j P_{21}\,|1{:}u_j \; 2{:}u_i\rangle$$

$$= b_j\,|1{:}u_i \; 2{:}u_j\rangle = B(2)\,|1{:}u_i \; 2{:}u_j\rangle$$

$\therefore \; P_{21} B(1) P_{21}^\dagger = B(2)$

Symmetric observable, e.g. $\vec{S} + \vec{I}$ commutes with P_{21}

3+ Particle Permutations on tensor product space $\bigotimes_{i=1}^{N} \mathcal{H}_i$

$$P_{npq}\,|1{:}u_i \;\; 2{:}u_j \;\; 3{:}u_k\rangle \equiv |n{:}u_i \;\; p{:}u_j \;\; q{:}u_k\rangle$$

(npq) = Permutation of (123) generalizes to N particles

P_{npq} places n^{th} object in place 1
" p^{th} " 2
" q^{th} " 3

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline n & p & q \\ \hline \end{array} \;\; P_{npq}$$

{Perms.} forms group G e.g. $P_{213}P_{132} = P_{312}$

$$\begin{array}{lccc} & 1 & 2 & 3 \\ P_{132} & 1 & 3 & 2 \\ P_{213} & 3 & 1 & 2 \end{array} \;\; P_{312}$$

6 permutations of 3 objects, $N!$ permutations of N objects $\mathcal{O}(G) = N!$

Transpositions: exchange 2 particles P_{132} P_{213} P_{321}

parity of perm = parity of # of transpositions

P_{213} and P_{132} are odd but $P_{312} = P_{213}P_{132}$ is even

Completely symmetric ket: $P_\alpha|\psi\rangle = |\psi\rangle \;\; \forall P_\alpha \in G \qquad |\psi\rangle \in \mathcal{H}_S$

Completely antisymmetric ket: $P_\alpha|\psi\rangle = \epsilon_\alpha|\psi\rangle \;\; \forall P_\alpha \in G \qquad |\psi\rangle \in \mathcal{H}_A$

$+1$ if α even
-1 if α odd

Group rearrangement theorem

$G = \{g_1\ g_2 \cdots g_N\}$ forms a group

Let $h \in G$ then $hG \equiv \{hg_1\ hg_2 \cdots hg_N\} = G$ rearranged in new order

proof if $hg_i = hg_j$ then $h^{-1}hg_i = h^{-1}hg_j \Rightarrow g_i = g_j \Rightarrow$ hG contains N distinct elements of G

$= G$